3-D co-geometry 2

1. Given the equation of a straight line is given as $2(x+2) = 2(y-3) = z+1$.

(a) Determine the vector equation of the straight line.

(b) Hence, find the coordinates of a point Q that lies on the straight line such that $|OQ| = 3\sqrt{2}$.

(a) $2(x+2) = 2(y-3) = z+1 \Rightarrow \frac{x+2}{1} = \frac{y-3}{1} = \frac{z+1}{2}$

The vector equation is $\boldsymbol{r} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$.

(b) Let $\mathrm{Q} = (-2+t, 3+t, -1+2t)$ be a point on the straight line.

$|OQ| = 3\sqrt{2} \Rightarrow OQ^2 = 18 \Rightarrow (-2+t)^2 + (3+t)^2 + (-1+2t)^2 = 18 \Rightarrow 6t^2 - 2t - 4 = 0$

$\Rightarrow 3t^2 - t - 2 = 0 \Rightarrow (3\mathrm{t}+2)(\mathrm{t}-1) = 0 \Rightarrow \mathrm{t} = -\frac{3}{2}, 1$

$\mathrm{Q} = \left(-\frac{7}{2}, \frac{3}{2}, -4\right)$ or $(-1,4,1)$.

2. Find the Cartesian equation of the plane passing through the points $A(4,-1,3)$ and $B(5,1,2)$ and containing the line : $\boldsymbol{r} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} + \lambda(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$.

$\boldsymbol{AB} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Direction of the given line is $\boldsymbol{v} = 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

Normal of the required plane is $\boldsymbol{N} = \boldsymbol{AB} \times \boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = \boldsymbol{i} + 2\boldsymbol{j} - 7\boldsymbol{k}$

Together with point $A(4,-1,3)$, the equation of the required plane is

$1(x-4) + 2(y+1) - 7(z-3) = 0$ or $\boldsymbol{x + 2y - 7z + 19 = 0}$

3. The position vectors of the points P, Q and R are $\boldsymbol{i} + 3\boldsymbol{k}, 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ respectively.

Let the plane π contains the points P, Q and R.

(a) Find a vector which is perpendicular to plane π.

(b) Find the area of ΔPQR.

(c) Obtain the Cartesian equation of plane π.

(a) $\overrightarrow{PQ} = \boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$, $\overrightarrow{PR} = -\boldsymbol{j} - 2\boldsymbol{k}$

$\boldsymbol{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -4 \\ 0 & -1 & -2 \end{vmatrix} = -8\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, which is a vector perpendicular to plane π.

(b) Area of $\Delta PQR = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}|-8\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}| = \frac{\sqrt{69}}{2}$

(c) The normal of the plane is $-8\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $P(1,0,3)$ is on the plane.
$\pi: -8(x-1) + 2(y-0) - (z-3) = 0$
$\pi: 8x - 2y + z = 11$

4. Find the vector $\boldsymbol{n}_1$ normal to the plane $\pi_1: \boldsymbol{r} = (5\boldsymbol{i} + \boldsymbol{j}) + \alpha(-4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) + \beta(\boldsymbol{j} + 2\boldsymbol{k})$.
Write down a vector $\boldsymbol{n}_2$ normal to the plane $\pi_2: 3x + y - z = 3$. Show that $4\boldsymbol{i} + 13\boldsymbol{j} + 25\boldsymbol{k}$ is normal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Given that the point $(1,1,1)$ lies on both π_1 and π_2.
Write down the equation of line of intersection of π_1 and π_2 in the form of $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ where t is a parameter.

The two directional vectors $\boldsymbol{d} = -4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{e} = \boldsymbol{j} + 2\boldsymbol{k}$ are on the plane, hence the normal is

$$\boldsymbol{n}_1 = \boldsymbol{d} \times \boldsymbol{e} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = -\boldsymbol{i} + 8\boldsymbol{j} - 4\boldsymbol{k}$$

$\boldsymbol{n}_2 = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

$$\boldsymbol{n}_2 \times \boldsymbol{n}_1 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & -1 \\ -1 & 8 & -4 \end{vmatrix} = 4\boldsymbol{i} + 13\boldsymbol{j} + 25\boldsymbol{k}$$

, thus $4\boldsymbol{i} + 13\boldsymbol{j} + 25\boldsymbol{k}$ is normal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

$\boldsymbol{n}_2 \times \boldsymbol{n}_1 = 4\boldsymbol{i} + 13\boldsymbol{j} + 25\boldsymbol{k}$ is a line parallel to the line of intersection of π_1 and π_2. Since $(1,1,1)$ lies on both π_1 and π_2, it must line on the intersection line.
Therefore, the equation of intersection line is $\boldsymbol{r} = (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + t(4\boldsymbol{i} + 13\boldsymbol{j} + 25\boldsymbol{k})$.

5. $\boldsymbol{OA} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{OB} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{OC} = 3\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{OD} = 2\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$.
Point P divides the line AC in the ratio 2 : 1 internally.
(a) Show that ABCD is a parallelogram.
(b) Calculate the exact area of the parallelogram ABCD.
(c) Find the position vector P and the angle APB in degrees corrected to one decimal place.

(a) $\boldsymbol{AB} = (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$
$\boldsymbol{DC} = (3\boldsymbol{i} + \boldsymbol{j}) - (2\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}) = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$
Since $\boldsymbol{AB} = \boldsymbol{DC}$, the opposite sides are equal and parallel, therefore ABCD is a parallelogram.

(b) $\boldsymbol{AC} = (3\boldsymbol{i} + \boldsymbol{j}) - (\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 2\boldsymbol{i} + 2\boldsymbol{k}$
Area of the parallelogram ABCD

$$= |\boldsymbol{AB} \times \boldsymbol{AC}| = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 3 \\ 2 & 0 & 3 \end{vmatrix} = |-6\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}| = \sqrt{(-6)^2 + 3^2 + 4^2} = \sqrt{61}.$$

(c) $\boldsymbol{OP} = \frac{(3\boldsymbol{i}+\boldsymbol{j})+2(\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})}{3} = \frac{1}{3}(5\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k})$

$\boldsymbol{PA} = (\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) - \frac{1}{3}(5\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) = \frac{1}{3}(-2\boldsymbol{i} - 2\boldsymbol{k})$

$\boldsymbol{PB} = (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) - \frac{1}{3}(5\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) = \frac{1}{3}(\boldsymbol{i} - 6\boldsymbol{j} + 7\boldsymbol{k})$

$\boldsymbol{PA} \cdot \boldsymbol{PB} = |\boldsymbol{PA}||\boldsymbol{PB}| \cos APB \Longrightarrow -\frac{16}{9} = \left(\frac{1}{3}\sqrt{8}\right)\left(\frac{1}{3}\sqrt{86}\right) \cos APB \Longrightarrow \cos APB = -\frac{16}{\sqrt{8}\sqrt{86}}$

$$angle\ APB = 127.6°$$

6. Show that the lines with equations

$\boldsymbol{r} = 7\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k} + \lambda(3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$ and $\boldsymbol{r} = 7\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k} + \mu(-2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ intersect, and find the position vector of their point of intersection.

The lines intersect if $7 + 3\lambda = 7 - 2\mu \Rightarrow 3\lambda = -2\mu \quad \ldots(1)$

$-3 - 2\lambda = -2 + \mu \Rightarrow \mu + 2\lambda = -1 \quad \ldots(2)$

$3 + \lambda = 4 - \mu \Rightarrow \mu + \lambda = 1 \quad \ldots(3)$

$(2) - (3),\ \lambda = -2 \ \ldots(4)$

$(4) \downarrow (3),\ \mu = 3 \ \ldots(5)$

Since (4), (5) satisfy (1), equations (1) – (3) has a unique solution : $\lambda = -2,\ \mu = 3$.

Therefore, the two given lines intersect.

The point of intersection is $7\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k} - 2(3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ or in Cartesian form (1, 1, 1).

7. Given a sphere $x^2 + y^2 + z^2 = 126$

(i) Find the equations of the tangent planes to the sphere when $x = 1$ and $y = 10$.

(ii) Find a point on the sphere that is farthest to the point (1, 2, 3). Hence, determine its distance.

(i) **Method 1**

When $x = 1$ and $y = 10$, $1^2 + 10^2 + z^2 = 126$, $z = -5$ or $z = 5$.

The points on the sphere are $A(1,10,-5)$, $B(1,10,5)$

For point $A(1,10,-5)$, the normal of the plane is $\mathbf{N}(1,10,-5)$

Hence, the equation of the plane is $1(x-1) + 10(y-10) - 5(z+5) = 0$

$$x + 10y - 5z = 126$$

For point $B(1,10,5)$, the normal of the plane is $\mathbf{N}(1,10,5)$

Hence, the equation of the plane is $1(x-1) + 10(y-10) + 5(z-5) = 0$

$$x + 10y + 5z = 126$$

Method 2 This method can be used for any surface other than sphere

Let $f(x,y) = z = \pm\sqrt{126 - x^2 - y^2}$

The equation of the tangent plane to the surface $z = f(x,y)$ at (x_0, y_0, z_0) is given by

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$$

Since $f_x = \mp\frac{x}{\sqrt{126-x^2-y^2}}$, $f_y = \mp\frac{y}{\sqrt{126-x^2-y^2}}$

The equation of the tangent plane at $A(1,10,-5)$ is

$$z + 5 = \frac{1}{\sqrt{126-1^2-10^2}}\,(x - 1) + \frac{10}{\sqrt{126-1^2-10^2}}\,(y - 10) \quad \text{or} \quad x + 10\,y - 5\,z = 126$$

The equation of the tangent plane at $A(1,10,5)$ is

$$z - 5 = -\frac{1}{\sqrt{126-1^2-10^2}}\,(x - 1) - \frac{10}{\sqrt{126-1^2-10^2}}\,(y - 10) \quad \text{or} \quad x + 10\,y + 5\,z = 126$$

(ii) The line joining the point (1, 2, 3) and the origin is

$$x = 1 + t, y = 2 + 2t, z = 3 + 3t$$

This line cuts the sphere, hence $(1 + t)^2 + (2 + 2t)^2 + (3 + 3t)^2 = 126$

$$t^2 + 2\,t + 1 = 9, \qquad t^2 + 2\,t - 8 = 0$$

$$t = -4 \text{ or } t = 2$$

When $t = -4$, $x = -3, y = -6, z = -9$.

Hence (-3, -6, -9) is a point on the sphere that is farthest to the point (1, 2, 3).

Its distance is $\sqrt{(1+3)^2 + (2+6)^2 + (3+9)^2} = \mathbf{4\sqrt{14}}$

Note that this distance is also the sum of the distances from (1, 2, 3) to origin and the radius of the sphere $=\sqrt{1+4+9} + \sqrt{126} = \mathbf{4\sqrt{14}}$

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